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On one connection between Lorentzian and Euclidean metrics

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Contents

Abstract

We investigate connections between pairs of (pseudo-)Riemannian metrics whose sum is a (tensor) product of a covector field with itself. A bijective mapping between the classes of Euclidean and Lorentzian metrics is constructed as a special result. The existence of such maps on a differentiable manifold is discussed. Similar relations for metrics of arbitrary signature on a manifold are considered. We point the possibility that any physical theory based on real Lorentzian metric(s) can be (re)formulated equivalently in terms of real Euclidean metric(s).

1. Introduction

In [1] the time is defined as a congruence of lines on a real differentiable manifold M. The vector field t tangent to this congruence is called *temporal* field. In the work mentioned is stated that the Maxwell equations on M with an Euclidean metric e_{ij}, $i, j = 1, \dots, n := \dim M$ are derivable from the standard electromagnetic Lagrangian on M with a (pseudo-)Riemannian metric $g_{ij} = t_i t_j - e_{ij}$, $t_i := e_{ij} t^j$. In the paper cited g_{ij} is said to be Lorentzian. Special metrics g_{ij} of this kind, when the norm of t is 2 (with respect to both metrics — see equation (4.9) below), are considered, e.g. in [2, sect. 2.6], [3, p. 219], and [4, p. 148, lemma 36]. A slight more general construction of the kind mentioned can be found in [3, pp. 241–242]. For it, without an investigation, is stated that it is Lorentzian again, which is not always the case (see Sect. 4 below). In the above constructions t can also be taken to be the gradient vector field of the global time function [5, 3].

The purpose of the present work is to be investigated pairs of (pseudo-)Riemannian metrics (g_{ij}, h_{ij}) whose sum is a product of the covariant components of some vector field t, i.e. $g_{ij} + h_{ij} = t_i t_j$ with, e.g., $t_i := g_{ij} t^j$.[1] In particular, we prove the important for the physics result that for any real Euclidean (resp. Lorentzian) metric there exists real Lorentzian (resp. Euclidean) metric forming with it such a pair.

In Sect. 2 we prove that if g_{ij} is an Euclidean metric, then (for $g_{ij} t^i t^j \neq 1$) the metric $h_{ij} = t_i t_j - g_{ij}$ can be only Lorentzian or negatively definite. As a corollary, we construct a map from the set of Euclidean metrics into the set of Lorentzian ones. To the applicability of the results of Sect. 2 is devoted Sect. 3. Here we point to some topological obstacles that may arise in this direction and formulate these results in a form of a proposition. The general case, for arbitrary (pseudo-)Riemannian metric g_{ij}, is investigated in Sect. 4. If g_{ij} has a signature (p, q), i.e. if the matrix $[g_{ij}]$ has p positive and q negative eigenvalues,[2] then the signature of h_{ij}, if it is non-degenerate, which is the conventional case, can be (q, p) or $(q + 1, p - 1)$. As a side-result, we obtain a map from the set of all Lorentzian metrics into the set of Euclidean ones. Some inferences of the results obtained are presented in Sect. 5. We construct bijective maps from the set of metrics with signature (p, q) on that with signature $(q + 1, p - 1)$, which, in particular, is valid for the classes of

[1] Bundle decompositions and correspondences between various types of metric tensors are consequences of the Witt (decomposition) theorem [6, chapter XIV, § 5]. The present paper deals with one specific such correspondence based on the use of a vector field t with appropriate properties.

[2] Some times the pair (p, q) is called type of g and the signature is defined as the number $s = p - q$. In this paper we suppose the numbers p and q to be independent of the point at which they are calculated, i.e. here we consider metrics whose signature is point-independent and so constant over the corresponding sets. The numbers p and q are also known as positive index and (negative) index of the metric. Often, especially in the physical literature, the signature is defined as an order n-tuple $(\varepsilon_1, \dots, \varepsilon_n)$ where p (resp. $q = n - p$) of $\varepsilon_1, \dots, \varepsilon_n$ are equal to $+1$ (resp. -1) or simply to the plus (resp. minus) sign and the order of $\varepsilon_1, \dots, \varepsilon_n$ corresponds to the one of the signs of the diagonal elements of the metric in some (pseudo-)orthogonal basis

Euclidean and Lorentzian metrics.[3] We also correct some wrong statements of [1]. Some concluding remarks are presented in Sect. 6. In particular, we construct, possibly under some conditions, bijective real maps between (pseudo-)Riemannian metrics of arbitrary signature.

Now, to fix the terminology, which significantly differs in different works, we present some definitions.

Following [8, p. 273], we call Riemannian metric on a real manifold M a non-degenerate, symmetric and 2-covariant tensor field g on it. If for any non-zero vector v at $x \in M$ is fulfilled $g_x(v, v) > 0$, the metric is called proper Riemannian, positive definite, or Euclidean; otherwise it is called indefinite or pseudo-Riemannian [8, 9]. It is known that every finite-dimensional paracompact differentiable manifold admits positively definite (Euclidean) metrics [9, chapter IV, § 1; chapter I, example 5.7], [10, chapter 1, excercise 2.3], [8, p. 280]. A pseudo-Riemannian metric with exactly one positive eigenvalue is called Lorentzian [2] (or, some times, Minkowskian).[4] If in the above definitions the non-degeneracy condition is dropped, the prefix 'semi-' is added to the names of the corresponding metrics [11]; e.g. a semi-Riemannian metric on M is a symmetric two times covariant tensor field on it [11].

2. Euclidean case

Let e be an Euclidean metric on a finite-dimensional, paracompact, differentiable, and real manifold M and t a vector field on M. Consider the 2-covariant symmetric tensor field

$$g = e(\cdot, t) \otimes e(\cdot, t) - e, \tag{2.1}$$

where $\otimes$ is the tensor product sign. In a local chart its local components are

$$g_{ij} = t_i t_j - e_{ij}, \qquad t_i = e_{ij} t^j. \tag{2.2}$$

Here and below the Latin indices run form 1 to $n := \dim M < \infty$ and a summation from 1 to n over indices repeated on different levels is assumed.

Let x be arbitrary point in M.

If $t|_x = 0$, then $g_{ij} = -e_{ij}$ and, consequently, g is at x a Riemannian metric with signature $(0, n)$ as that of e is $(n, 0)$.

If $t|_x \neq 0$, then we can construct by means of a Gramm-Schmidt procedure [12, chapter 4, § 3], [13, pp. 206–208] an orthonormal (with respect to e) basis $\{E_i\}$ in the tangent to M space at x with $E_1 = \frac{1}{\alpha} t$ where $\alpha = +\sqrt{e(t, t)} > 0$. With respect to such a basis we have $e_{ij} = \delta_{ij}$, $t^i = \alpha\delta^i_1$, and $t_i = e_{ij} t^j = \alpha\delta_{i1}$ where $\delta_{ij} := \delta^j_i := \delta^{ij}$ are the Kroneker symbols, i.e.

[3] A (partial) correspondence between Euclidean and Lorentzian metrics is established in [7] via the Einstein equations.

[4] One can also find the definition of a Lorentzian metric as a metric with only one negative eigenvalue [4, p. 55]. This definition is isomorphic to the one used in the present paper (see, e.g., [4, pp.92–93].

$\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. In the basis mentioned the matrix of g is diagonal and

$$[g_{ij}] = \operatorname{diag}\big(e(t,t) - 1, -1, \dots, -1\big) \tag{2.3}$$

due to (2.2). Therefore g is a Riemannian metric on M if and only if the Euclidean norm of t is not equal to one at every point, i.e.

$$e(t,t) \neq 1. \tag{2.4}$$

From (2.3) we see that the first eigenvalue of g is $\lambda_1 \gtreqless 0$ iff $e(t,t)|_x \gtreqless 1$, the remaining $n-1$ eigenvalues being equal to -1. Since the metric's signature is independent of the local coordinates by means of which it is computed,[5] from here an important result follows: if $e(t,t) > 1$ at every point, the metric g is Lorentzian, if $e(t,t) < 1$ at every point, it is negatively definite and, hence, isomorphic to an Euclidean one, and for $e(t,t) = 1$ at every point it is semi-Riemannian with signature $(0, n-1, 1)$.

Summing up, if there exists a vector field t satisfying (2.4) or $e(t,t) = 1$ at every point, then (2.1) defines a metric on M for which there are three possibilities: First, if $e(t,t) > 1$, it is Lorentzian. Second, if $e(t,t) = 1$, it is semi-Riemannian, viz. a 1-degenerate metric, and, consequently, non-Riemannian one [11]. And third, if $e(t,t) < 1$, it is negatively definite, and so isomorphic to an Euclidean metric. From physical view-point, the most essential result is that if for every e we choose some vector field t_e with $e(t_e, t_e) > 1$, then the mapping $e \mapsto g$, given by (2.1) for $t = t_e$, maps the class of Euclidean metrics on M into the class of Lorentzian ones. It is clear, this map essentially depends on the choice of the vector fields t_e used in its construction.

3. Applicability of the results

Up to this point we have supposed two major things: the existence of an Euclidean metric e and of a vector field t with the corresponding properties on the *whole* manifold M. In this sense the above considerations are global. Of course, we can localize them by restricting ourselves on some subset U of M. Different conditions for global or local existence of (Euclidean) metrics are well-known and discuss at length in the corresponding literature (see, e.g., [14, chapter IV] or [9, 15]). In our case, the existence of Euclidean metric on M is a consequence of the paracompactness and finite-dimensionality of the manifold M [9]. These assumptions are enough for the most physical applications and we assume they are valid in this work.[6]

What concerns the existence of a vector field t with properties required ($e(t,t)$ to be greater than, or equal to, or less than one), some problems may arise. If on t we do not impose additional restrictions, it always can

[5] This is a trivial corollary of the results of [6, chapter XIV, §7].
[6] See the partial discussion of this problem in [3, sect. 5.2].

be constructed as follows: Take a non-vanishing on M vector field t_0,[7] so $e(t_0, t_0) \neq 0$ (everywhere on M). Defining $t := \sqrt{a} t_0 / \sqrt{e(t_0, t_0)}$ for $a \in \mathbb{R}$, $a \geq 0$, we get $e(t, t) = a$. Hence, choosing $a \gtreqqless 1$, we obtain $e(t, t) \gtreqqless 1$. Obviously, the existence of t in the first two cases, $e(t, t) \geq 1$, is equivalent to the existence of a non-vanishing vector field on M, while in the last one, $e(t, t) < 1$, this is not necessary, viz. in it t may vanish on some subsets on M or even to be the null vector field on M.

The general conclusion is: a vector field the with $e(t, t) \geq 1$ (over M) exits iff M admits nowhere vanishing (on M) vector field. Thus, our results concerning the case $e(t, t) \geq 1$ are applicable iff such a field exists. As we said above, this is just the situation if we do not impose additional conditions on t. But this is not satisfactory from the view-point of concrete applications. For instance, in the most mathematical investigations the (Euclidean or (semi-)Riemannian) metrics are required to be differentiable of class C^1 [9, 15, 16, 8]; e.g. in the Riemannian geometry one normally uses C^2 metrics. Such an assumption implies t to be of class of smoothness at least C^1. Analogous is the situation in physics, for example, the treatment of t as a temporal field requires t to be at least continuous [1] and the considerations on the background of general relativity force us to assume t to be of class C^2 [2].

Therefore of great importance is the case when the vector field t satisfies certain smoothness conditions, viz. when it is of class C^m for some $m \geq 0$. At this point some topological obstacles may arise for the global existence of t with $e(t, t) \geq 1$. In fact, the above-said implies that a vector field t of class C^m with $e(t, t) \geq 1$ exists on $U \subseteq M$ iff on U there exists a C^m non-vanishing vector field. But it is well-known that not every manifold admits such a tangent vector field [17]. A classical example of this kind are the even-dimensional spheres $\mathbb{S}^{2k}$, $k \in \mathbb{N}$: on $\mathbb{S}^{2k}$ does not exist non-vanishing (on the whole $\mathbb{S}^{2k}$) continuous vector field [17], [18, sect. 4.24]. Examples of the opposite kind are the odd-dimensional spheres $\mathbb{S}^{2k-1}$ [17], [18, excerise. 4.26] and the path-connected manifolds with flat C^1 linear connection: they always admits global C^1 non-vanishing vector fields.[8] Also every non-compact manifold admits C^0 non-zero vector field [19]. An analysis of the question of existence of vector fields (and Lorentz metrics) can be found in [4] where also other examples are presented. Consequently, the global existence of C^m, $m \geq 0$ field t with $e(t, t) \geq 1$ depends on the concrete manifold M and has to be investigated separately for any particular case.

In conclusion, the results of the preceding section are valid locally and for their global, i.e. on the whole manifold M, validity may arise obstacles of pure topological nature. Since on M, due to the paracompactness and finite dimensionality, an Euclidean metric always exists, this is connected

[7] Generally t_0 is discontinuous (see below).

[8] In the last case such a vector field can be constructed as follows. Fix a non-zero vector v_0 at an arbitrary point $x_0 \in M$. Define the vector field v at any $x \in M$ as the result of the parallel transport, assigned to the given flat connection, of v_0 from x_0 to x along some path connecting x_0 and x. Then v is a tangent vector field on M which is non-vanishing and of class C^1.

with the existence of a vector field t with properties required. So, the precise formulation of the results obtained is the following.

Proposition 3.1. *Let t be a vector field over $U \subseteq M$, e be an Euclidean metric on U, and $U_{\gtreqless} := \{x \mid \ x \in U,\ e(t,t)|_x \gtreqless 1\}$. Then the tensor field* (2.1) *is:*

(i) *a negatively definite Riemannian metric on $U_<$;*

(ii) *a Lorentzian metric on $U_>$;*

(iii) *a 1-degenerate negatively definite semi-Riemannian metric on $U_=$.*

The most interesting is the 'smooth' global case when $U = M$, two of the sets $U_<$, $U_>$, and $U_=$ are empty, and t is of class C^m, $m \geq 0$. As we notice above, one can always choose t such that $U_> = U_= = \varnothing$ and $U_< = M$ but the two other cases, $U_> = M$ and $U_= = M$, can not be realized for arbitrary manifold M if t is of class C^m with $m \geq 0$. If we drop the smoothness requirement, then t can always be chosen such that one of the sets $U_<$, $U_>$, and $U_=$ to be equal to M, the other two being the empty set.

4. General case

It is said that a Riemannian metric g on $U \subseteq M$ is of signature (p, q), $p + q = n := \dim M$, if it has p positive and q negative eigenvalues. A semi-Riemannian metric on U is of signature (p, q) and defect r (or of signature (p, q, r), or r-degenerate with signature (p, q)), $p + q + r = n$, if it has p positive, q negative, and r vanishing eigenvalues.

Proposition 4.1. *Let g be a Riemannian metric of signature (p, q) on $U \subseteq M$, t be a vector field on U, $\widetilde{U}^{+}_{\gtreqless} := \{x \mid\ x \in U,\ g(t,t)|_x \gtreqless 1\}$, and*

$$g \mapsto \tilde{g}^{+} := h := g(\cdot, t) \otimes g(\cdot, t) - g. \tag{4.1}$$

Then the tensor field h is:

(i) *a Riemannian metric with signature (q, p) on $\widetilde{U}^{+}_{<}$.*

(ii) *a Riemannian metric with signature $(q + 1, p - 1)$ on $\widetilde{U}^{+}_{>}$.*

(iii) *a (parabolic) semi-Riemannian metric with signature $(q, p - 1)$ and defect 1 on $\widetilde{U}^{+}_{=}$, i.e. on $\widetilde{U}^{+}_{=}$ the bilinear map h has q positive, $(p-1)$ negative, and 1 vanishing eigenvalue.*

Proof. Since g is by definition 2-covariant symmetric tensor field, such is h too. So, it remains to be studied the eigenvalues of h.

Let $x \in U$ be arbitrarily fixed point. We shall prove the proposition at x, i.e. for $U = \{x\} \subset M$. Then the general result will be evident as $U = \bigcup_{x \in U} \{x\}$. All of the bellow written quantities in this proof will be taken at x; so their restriction at x will not be written explicitly. We shall distinguish three cases.

'Null' case, $t = 0$. The statement is evident as now $g(t, t) = 0 < 1$ and $h = -g$; so the signature of h is (q, p).

'Non-isotropic' case, t is non-isotropic, i.e. $g(t,t) \neq 0$ and hence $t \neq 0$. Let $\{E'_i\}$ be a basis in $T_x(M)$, the space tangent to M at x, and consisting of non-isotropic vectors with $E_1 = t$. Applying to this basis the standard Gramm-Schmidt orthogonalization procedure [12, chapter 4, § 3], [13, pp. 206–208], with respect to the scalar product $(\cdot,\cdot) = g(\cdot,\cdot)$, we can construct (after normalization) a (pseudo-)Riemannian basis $\{E_i\}$ (at x) such that $E_1 = t/\alpha$, $\alpha := +\sqrt{|g(t,t)|}$ and $g_{ij} := g(E_i,E_j) = \varepsilon_i\delta_{ij}$ (i is not a summation index here!) where $p \in \mathbb{N}\bigcup\{0\}$ of the numbers $\varepsilon_1,\dots,\varepsilon_n$ are equal to $+1$ while the others $q = n - p$ of them are equal to -1 and δ_{ij} are the Kroneker deltas. With respect to $\{E_i\}$, we easily obtain

$$[h_{ij}] = \operatorname{diag}\big(\varepsilon_1(g(t,t)-1), -\varepsilon_2,\dots,-\varepsilon_n\big), \qquad \varepsilon_1 = \operatorname{sign}(g(t,t)). \tag{4.2}$$

From here the formulated results follow immediately.

'General isotropic' case, t is non-zero and isotropic, i.e. $t \neq 0$ and $g(t,t) = 0$. As it is easily seen, this is possible only for $n := \dim M \geq 2$ and $q,p \geq 1$. Now the above method can not be applied directly because the Gramm-Schmidt procedure fails if some of the initial vectors are/is isotropic (cf., e.g., its construction in [13, pp. 206–208]). Let $\{E''_i\}$ be some fixed basis in which the components of g are $g''_{ij} = \varepsilon''_i\delta_{ij}$, $\varepsilon''_i = \pm 1$. Since $t \neq 0$ and $0 = g(t,t) = \sum_i \varepsilon''_i (t''^i)^2$, at least two of the components t''^i of t in $\{E''_i\}$ are non-null. Let these two non-zero components correspond to $i = 1,2$, which can be achieved by an appropriate renumbering of the initial basis. Let $\{E'_i\}$ be a basis of non-isotropic vectors such that E'_1 and E'_2 are fixed and in $\{E''_i\}$ their components are

$$(E'_k)''^1 = \frac{1}{a}\, t''^1\, \delta_{k1}, \qquad (E'_k)''^i = \frac{1}{a}\, t''^i\, \delta_{k2}, \qquad \text{for } k = 1,2 \text{ and } i \geq 2,$$

where $a := +\sqrt{|\varepsilon''_1 (t''^1)^2|} = |t''^1| = +\Big|\sum_{i\geq 2}\varepsilon''_i (t''^i)^2\Big|^{1/2} > 0$ due to $0 = g(t,t) = \sum_i \varepsilon''_i (t''^i)^2$ and $t''^1, t''^2 \neq 0$. A simple computation in $\{E''_i\}$ shows that $t''^i = a\big((E'_1)''^i + (E'_2)''^i\big)$ and $g(E'_i,E'_j) = \varepsilon''_i\delta_{ij}$ for $i,j = 1,2$. So E'_1 and E'_2 are non-isotropic and $t = a(E'_1 + E'_2)$. Now applying to $\{E'_i\}$ the Gramm-Schmidt procedure (for the scalar product $(\cdot,\cdot) = g(\cdot,\cdot)$) and normalizing the vectors, we get a (pseudo-)orthogonal basis $\{E_i\}$ such that

$$g_{ij} := g(E_i,E_j) = \varepsilon_i\delta_{ij}, \qquad E_1 = E'_1, \quad E_2 = E'_2, \quad \text{and} \quad t = a(E_1 + E_2)$$

where p of $\varepsilon_1,\dots,\varepsilon_n$ are equal to $+1$ and $q = n - p$ of them are equal to -1 (and also $\varepsilon_1 = \varepsilon''_1$, $\varepsilon_2 = \varepsilon''_2$). Since in $\{E_i\}$ the components of t are $t^i = a(\delta^{i1} + \delta^{i2})$, we have $t_i = g_{ij}t^j = a\varepsilon_i(\delta_{i1} + \delta_{i2})$. The equality $\varepsilon_1 + \varepsilon_2 = 0$ is also valid by virtue of $g(t,t) = 0$ and $a \neq 0$. Therefore in $\{E_i\}$ the matrix of h is

$$[h_{ij}] = \operatorname{diag}\left(\begin{pmatrix} a^2 - \varepsilon_1 & a^2\varepsilon_1\varepsilon_2 \\ a^2\varepsilon_1\varepsilon_2 & a^2 - \varepsilon_2 \end{pmatrix}, -\varepsilon_3,\dots,-\varepsilon_n\right) \tag{4.3}$$

Consequently, due to $\varepsilon_1 + \varepsilon_2 = 0$ and $\varepsilon_1\varepsilon_2 = -1$, the eigenvalues of $[h_{ij}]$, which are the roots of $\det[h_{ij} - \lambda\delta_{ij}] = 0$, are $\lambda_i = -\varepsilon_i$ for $i \geq 3$ and $\lambda_\pm = a^2 \pm$

$\sqrt{1+a^4} \gtrless 0$. So, in this case the signature of h is $(q-1, p-1)+(1,1) = (q,p)$ as that of g is (p,q). □

Corollary 4.1. *Let g be a Riemannian metric of signature (p,q) on $U \subseteq M$ and t be a vector field on U. Assume t can be chosen such that $g(t,t)$ is less than, or greater than, or equal to one on the whole set U. Then on U the tensor field h given by* (4.1) *is:*

(i) *a Riemannian metric with signature (q,p) for $g(t,t) < 1$.*

(ii) *a Riemannian metric with signature $(q+1, p-1)$ for $g(t,t) > 1$.*

(iii) *a (parabolic) semi-Riemannian metric with signature $(q, p-1)$ and defect 1 for $g(t,t) = 1$, i.e. in this case h has q positive, $(p-1)$ negative, and 1 vanishing eigenvalue.*

Proof. This result is a version of proposition 4.1 corresponding to the choice of t such that one of the sets $\widetilde{U}^{+}_{<}$, $\widetilde{U}^{+}_{>}$, and $\widetilde{U}^{+}_{=}$ is equal to U. □

Evidently, the main results formulated in sections 2 and 3 are special cases, corresponding to $(p,q) = (n,0)$, of the just proved proposition 4.1 and corollary 4.1.

It is clear that if g is a Riemannian metric on U, then, choosing arbitrary some vector field t on U with $g(t,t) > 1$, the map (4.1) yields (infinitely) many (semi-)Riemannian metrics on U whose signature (and, possibly, defect) depends on the norm $g(t,t)$ on U. Generally different t generate different metrics $\tilde{g}^{+}$ from one and the same initial metric g.

At this point a natural question arises: Does a vector field t with the properties described in corollary 4.1 exist? Here, when $U = M$, the situation is completely the same as described in Sect. 3 for the Euclidean case ($g = e$). If on t are not imposed some additional, e.g. smoothness, conditions, a vector field t on U with $g(t,t)|_U \gtreqless 1$ can always be constructed for every $U \subseteq M$. In fact, let t_0 be any (generally discontinuous) non-vanishing on U vector field. By rescaling locally the components of t_0 we can obtain from it a non-vanishing vector field t'_0 such that $g(t'_0, t'_0)|_U \neq 0$ and $\operatorname{sign}(g(t'_0, t'_0)|_U) = \varepsilon = \text{const}$. Defining $t := \sqrt{a} t'_0 |g(t'_0, t'_0)|^{-1/2}$ for $a \in \mathbb{R}$, $a \geq 0$, we get $g(t,t) = \varepsilon a$. Consequently, by an appropriate choice of ε and a, we can realize t with $g(t,t)|_U \gtreqless 1$. Since g is by definition nondegenerate (the kernel of g consists of the null vector field on U), the relation $g(t,t)|_U \geq 1$ implies t to be non-vanishing vector field. Obviously, this conclusion does not concern the case of t with $g(t,t)|_U < 1$ when t can vanish somewhere or everywhere on U.

As we know from Sect 3, the situation for t with $g(t,t)|_U \geq 1$ is completely different when $U = M$ and C^m, $m \geq 0$, metrics and vector fields are considered: Generally such vector field does not exist globally, i.e. on the whole manifold M. This existence depends on the topological properties of M and has to be investigated separately in any particular case.

Now consider the class of (resp. smooth) Lorentzian metrics on M, i.e. those g for which $(p,q) = (1, n-1)$. For them, according to corollary 4.1, the metric h is of signature $(n-1, 1)$ for $g(t,t) < 1$ and $(n,0)$ for $g(t,t) > 1$ (resp. if such t exists on M), i.e. in the former case g and h are isomorphic

and in the latter one h is Euclidean metric. Thus, if for every g we choose some vector field t_g with $g(t_g, t_g) > 1$, then the whole class of (resp. smooth) Lorentzian metrics is mapped into the class of (resp. smooth) Euclidean ones by the mapping $g \mapsto h$ given by (4.1) for $t = t_g$ (resp. if such smooth t_g exists on M). Evidently, different vector fields t_g realize different such maps.

Now some natural questions are in order. Let G^U (resp. $G^U_{p,q}$) be the set of all Riemannian metrics (resp. of signature (p, q)) on $U \subseteq M$. If t is a fixed vector field on U, then what is the character of the map $\varphi^t_U \colon G^U \to G^U$ given by (4.1)? For instance, can it be surjective, injective, or bijective? Can any two Riemannian metrics (with 'corresponding' signatures) be mapped into each other by φ^t_U for a suitable t? Etc.

Proposition 4.2. *Let $g \in G^U_{p,q}$, t be arbitrarily fixed vector field on U, and $\varphi^t_U \colon G^U \to G^U$ be given via (4.1). Then:*

(i) *The map $\varphi^t_U\big|_{t=0}$ is bijection.*

(ii) *If $t \neq 0$, the map φ^t_U is injection on the sets $\{g : g \in G^U_{p,q}, \quad g(t,t) = 1/2\}$ and $\{g : g \in G^U_{p,q}, \quad g(t,t) = 0\}$.*

(iii) *If $g(t,t) \neq 0, \frac{1}{2}$, then there exists a metric $g' \in G^U_{p,q}$ such that $\varphi^t_U(g) = \varphi^t_U(g')$ and $g \neq g'$. Besides, this g' is unique iff $n := \dim M = 1$ or $n \geq 2$ and $g(t,t) \neq 1$, i.e. φ^t_U is two-to-one mapping on the sets $\{g : g \in G^U_{p,q}, \quad g(t,t) \neq 0, \frac{1}{2}\}$ for $n = 1$ and $\{g : g \in G^U_{p,q}, \quad g(t,t) \neq 0, \frac{1}{2}, 1\}$ for $n \geq 2$. For $n \geq 2$ and $g(t,t) = 1$ this metric g' depends on $(n-1)$ non-zero real parameters.*

Proof. **Case (i)**. For $t = 0$ we have $\varphi^t_U(g) = -g$, so φ^0_U is reversing of the the metric [4, p. 92] and hence it is bijective.

Cases (ii) for $g(t,t) = 1/2$ and (iii). Consider the equation $\varphi^t_U(g) = \varphi^t_U(g')$, $t \neq 0$ with respect to g'. In the special basis (at some $x \in U$ and with respect to g) defined in the 'non-isotropic' case of the proof of proposition 4.1, this equation reads

$$\alpha^2(\delta_{i1}\delta_{j1} - g'_{i1}g'_{j1}) = \varepsilon_i\delta_{ij} - g'_{ij}, \qquad \varepsilon_i = \pm 1, \quad \alpha \neq 0. \tag{4.4}$$

In the same basis we also have $g_{11} = \varepsilon_1$ and $g(t,t) = \varepsilon_1\alpha^2$.

At first we consider the one-dimensional case, $n := \dim M = 1$. Now, multiplying (4.4) with ε_1, we get $g(t,t)(1 - \varepsilon_1 g'_{11})(1 + \varepsilon_1 g'_{11}) = 1 - \varepsilon_1 g'_{11}$. So, the last equation has two solutions with respect to g'_{11}: $g'_{(1)\,11} = \varepsilon_1 = g_{11}$ and $g'_{(2)\,11} = \big(\frac{1}{g(t,t)} - 1\big)\varepsilon_1 = \big(\frac{1}{g(t,t)} - 1\big)g_{11}$ By virtue of $n = 1$, this means that only two metrics g' satisfy (4.4): $g'_{(1)} = g$ and $g'_{(2)} = \big(\frac{1}{g(t,t)} - 1\big)g$.

Evidently $g'_{(2)} \neq g$ iff $g(t,t) \neq 1/2$ which completes the proof for $n = 1$ of the cases (ii) for $g(t,t) = 1/2$ and (iii).

Let now $n \geq 2$.

For $i = j = 1$ equation (4.4) yields

$$\alpha^2\big(1 - (g'_{11})^2\big) = \varepsilon_1 - g'_{11} \tag{4.5}$$

with solutions

$$g'_{(1)\,11} = \varepsilon_1 = g_{11} \qquad \text{and} \qquad g'_{(2)\,11} = \left(\frac{1}{g(t,t)} - 1\right) g_{11}. \tag{4.5$'$}$$

For $i, j \geq 2$ equation (4.4) reduces to

$$-\alpha^2 g'_{i1} g'_{j1} = \varepsilon_i \delta_{ij} - g'_{ij}, \qquad i, j \geq 2. \tag{4.6}$$

At last, for $i \neq 1$, $j = 1$ or $i = 1$, $j \neq 1$ equation (4.4) gives $\alpha^2 g'_{i1} g'_{11} = g'_{i1}$, the solutions of which are

$$g'_{i1} = 0 \quad (= g_{i1}), \qquad i \geq 2 \tag{4.7a}$$

and

$$g'_{11} = \frac{1}{\alpha^2}, \quad g'_{i1} \neq 0 \quad \text{for } i \geq 2. \tag{4.7b}$$

Consider the case (4.7a). Now (4.6) reads $g'_{ij} = \varepsilon_i \delta_{ij} = g_{ij}$ for $i, j \geq 2$. So (see (4.7a)) $g'_{ij} = g_{ij}$ for $(i, j) \neq (1, 1)$. Combining this with (4.5′), we find the following two solutions for g':

$$g'_{(1)} = g \quad \text{and} \quad g'_{(2)\,ij} = \begin{cases} g_{ij} & \text{for } (i,j) \neq (1,1) \\ \big(\frac{1}{g(t,t)} - 1\big) g_{11} & \text{for } (i,j) = (1,1) \end{cases} .$$

Evidently $g'_{(2)} \neq g$ iff $g(t, t) \neq 1/2$ which proves the cases under consideration for $g(t, t) = 1/2$ and $n \geq 2$.

The proof of the rest of case (iii) is a consequence of case (4.7b). Substituting (4.7b) into (4.5), we get $\alpha^2 = \varepsilon_1$ and, consequently, as $\alpha \in \mathbb{R}$, this is possible if and only if $g(t, t) = 1 = \varepsilon_1^2 = \alpha^2$. If this is the case, equation (4.6) yields $g'_{ij} = \varepsilon_i \delta_{ij} + g'_{i1} g'_{j1}$, $i, j \geq 2$. Combining the last results with (4.7b), we obtain for $g(t, t) = 1$ the solution

$$g'_{(3)\,ij} = \begin{cases} g_{11} = 1 & \text{for } i, j = 1 \\ g_{ij} + a_i a_j & \text{for } i, j \geq 2 \\ a_i & \text{for } i \geq 2,\ j = 1 \\ a_j & \text{for } i = 1,\ j \geq 2 \end{cases}$$

where a_i, $i \geq 2$ are arbitrary non-zero real numbers. As $a_i \neq 0$ (see (4.7b)), we have $0 \neq g'_{(3)\,i1} \neq g_{i1} = 0$ for $i \geq 2$. Therefore, if $\dim M \geq 2$ and $g(t, t) = 1$, the initial equation admits an $(n - 1)$ parameter family of solutions $g' \neq g$

At last, we have to consider the **case (ii) for** $g(t, t) = 0$ **with** $t \neq 0$. This is possible only for $n \geq 2$ (see the last part of the proof of proposition 4.1).

In the special basis $\{E_i\}$ introduced in the 'general isotropic' case of the proof of proposition 4.1, the equation $\varphi^t_U(g) = \varphi^t_U(g')$, after some algebra, takes the form

$$a^2 \big[\varepsilon_i \varepsilon_j (\delta_{i1} + \delta_{i2})(\delta_{j1} + \delta_{j2}) - (g'_{i1} + g'_{i2})(g'_{j1} + g'_{j2})\big] = \varepsilon_i \delta_{ij} - g'_{ij}. \tag{4.8}$$

For $i,j=1,2$ this equation reduces to

$$a^2\big[1+(g'_{11}+g'_{12})(g'_{22}+g'_{12})\big]=g'_{12}, \qquad a^2\big[1-(g'_{ii}+g'_{12})^2\big]=\varepsilon_i-g'_{ii}$$

as $\varepsilon_1+\varepsilon_2=0$ and $\varepsilon_i^2=1$. These equations form a closed system for g'_{ij} with $i,j=1,2$. It has a unique solution. Actually, putting for brevity $x_i:=g'_{i1}+g'_{i2}$, $i=1,2$ and $z:=g'_{12}$, we see that the equations mentioned can be written as $z=a^2(1+x_1x_2)$ and $a^2x_i^2-x_i+\varepsilon_i-a^2+z=0$, $i=1,2$ and hence $a^2x_i^2-x_i+\varepsilon_i-a^2x_1x_2=0$, $i=1,2$. Forming the sum and difference of the last two equations, we, by virtue of $\varepsilon_1+\varepsilon_2=0$, find $a^2(x_1+x_2)^2-(x_1+x_2)=0$ and $a^2(x_1+x_2)(x_1-x_2)-(x_1-x_2)+\varepsilon_1-\varepsilon_2=0$ respectively. The first of these equations has two solutions, $(x_1+x_2)=0$ and $(x_1+x_2)=1/a^2$, but only the first of them, $(x_1+x_2)=0$, is compatible with the second of the above equations due to $\varepsilon_1+\varepsilon_2=0$, $\varepsilon_{1,2}=\pm1$. In this way we see that $(x_1+x_2)=0$ and $(x_1-x_2)=\varepsilon_1-\varepsilon_2$. Therefore we obtain $x_1=\varepsilon_1$, $x_2=\varepsilon_2$, and $z=a^2(1+\varepsilon_1\varepsilon_2)=0$. Returning to the initial variables, we, finally, get

$$g'_{ij}=g_{ij}=\varepsilon_i\delta_{ij} \quad \text{for } i,j=1,2.$$

Consider now (4.8) for $i=1,2$ and $j\ge3$: $a^2(g'_{i1}+g'_{i2})(g'_{j1}+g'_{j2})=g'_{ij}$. By virtue of the last result it reduces to $a^2\varepsilon_i(g'_{j1}+g'_{j2})=g'_{ij}$, $i=1,2$, $j\ge3$. Summing the equations corresponding to $i=1,2$ and using $\varepsilon_1+\varepsilon_2=0$, we find $g'_{1j}+g'_{2j}=0$ for $j\ge3$, which, when inserted into the initial equations, leads to

$$g'_{ij}=g_{ij}=0, \qquad i=1,2, \quad j\ge3.$$

At last, for $i,j\ge3$ equation (4.8) gives $-a^2(g'_{i1}+g'_{i2})(g'_{j1}+g'_{j2})=\varepsilon_i\delta_{ij}-g'_{ij}$. Substituting here the last result, we find

$$g'_{ij}=g_{ij}=\varepsilon_i\delta_{ij} \qquad \text{for } i,j\ge3.$$

Combining the last three results, we, finally, obtain $g'=g$, i.e. the only solution of (4.8) is $g'=g$. Hence, in the case considered the map φ_U^t is an injection. $\square$

Remark 4.1. From the proof of proposition 4.2 follows that in the first subcase of case (iii) of proposition 4.2, when φ_U^t is 2:1 map, is fulfilled $g'(t,t)=1-g(t,t)$ (with $g'=g'_{(2)}$) while in the second subcase of case (iii) is valid $g'(t,t)=g(t,t)$ (with $g'=g'_{(3)}$).

Proposition 4.3. *Let the vector field t be arbitrarily fixed on U and the map $\varphi_U^t\colon G^U\to G^U$ be given by (4.1). Then $(\varphi_U^t\circ\varphi_U^t)(g)=g$ iff $g\in G^U$ is such that $g(t,t)=0,2$.*

Remark 4.2. Note, due to (4.1), we have

$$\big(\varphi_U^t(g)\big)(t,t)=g(t,t) \qquad \text{iff } g(t,t)=0,2. \tag{4.9}$$

Proof. Consider the case $g(t,t) \neq 0$. In a basis in which (4.2) holds, equation (4.1) gives ($h := \varphi_U^t(g)$)

$$\big[(\varphi_U^t(h))_{ij}\big] = \operatorname{diag}\big(\varepsilon_1[g(t,t)-1][g(t,t)(g(t,t)-1)-1], \varepsilon_2, \dots, \varepsilon_n\big)$$

because of $t^i = \alpha\delta^{i1}$, $g(t,t) = \varepsilon_1\alpha^2$, $t_i^g := g_{ij}t^j = \varepsilon_1\alpha\delta_{i1}$, and $t_i^h := h_{ij}t^j = \varepsilon_1\alpha(g(t,t)-1)\delta_{i1}$. A simple computation shows that the last matrix is equal to $g_{ij} = \operatorname{diag}(\varepsilon_1,\dots,\varepsilon_n)$ iff $g(t,t) = 2$ as we supposed $g(t,t) \neq 0$.

If $t = 0$, then $(\varphi_U^0 \circ \varphi_U^0)(g) = +\varphi_U^0(-g) = g$ for every $g \in G^U$ due to (4.1).

At last, let $t \neq 0$ and $g(t,t) = 0$. In a basis in which (4.3) holds, from (4.1), we obtain $[(\varphi_U^t(h))_{ij}] = [g_{ij}]$ due to $t^i = a(\delta^{i1}+\delta^{i2})$, $t_i^g := g_{ij}t^j = a\varepsilon_i(\delta_{i1}+\delta_{i2})$, and $t_i^h := h_{ij}t^j = -a(\varepsilon_1\delta_{i1} + \varepsilon_2\delta_{i2})$. □

From the just proved result immidiately follows (see also [20, p. 14, proposition 6.9])

Corollary 4.2. *The map φ_U^t for given t is bijective on the sets $G_{t;2}^U := \{g : g \in G^U,\ g(t,t) = 2\} \subset G^U$ and $G_{t;0}^U := \{g : g \in G^U,\ g(t,t) = 0\} \subset G^U$.*

Remark 4.3. The bijectiveness of φ_U^t on $G_{t;2}^U$ does not contradict to proposition 4.2, case (ii). Actually, if $g \in G_{t;2}^U$, $g' \in G^U$, $g \neq g'$, and $\varphi_U^t(g') = \varphi_U^t(g)$, then (see remark 4.1) $g'(t,t) = 1 - g(t,t) = -1 \neq 2$, i.e. g' is not in $G_{t;2}^U$.

Ending this section, we have to note that if the Riemannian metrics g and h are given, then generally there does not exists a vector field t connecting them through $h = g(\cdot,t) \otimes g(\cdot,t) - g$. There are two reasons for this. On one hand, by propositon 4.1 for this the metrics g and h must be of 'corresponding' signature, viz. (p,q) and $(q+1,p-1)$ or (p,q) and (q,p) respectively. On the other hand, in local coordinates the mentioned connection between g and h reduces to a system of $n(n+1)/2$ equations for the n components of t and, consequently, it has solution(s) only in some exceptional cases. It is clear that even for Euclidean metric g and Lorentzian metric h such t exixts only as an exception, not in the general case.

5. Consequences

Since the case (ii) of corollary 4.1 is most important in connection with possible applications, we investigate below some consequences of it.

Let $G_{p,q}^U$ be the set of all Riemannian metrics with signature (p,q) on $U \subseteq M$.

Corollary 5.1. *Let $p \geq 1$, for every $g \in G_{p,q}^U$ be chosen a vector field t_g on U such that $g(t_g,t_g) = 2$, and $T := \{t_g : g \in G_{p,q}^U\}$. Then the map $\varphi_{p,q}^T \colon G_{p,q}^U \to G_{q+1,p-1}^U$ given via*

$$g \mapsto \tilde{g}^+ := g(\cdot,t_g) \otimes g(\cdot,t_g) - g \tag{5.1}$$

is bijective, i.e. one-to-one onto map.

Proof. At first we note that t_g with $g(t_g, t_g) = 2$ always exists for every g because of $p \geq 1$. (E.g. one can set $t_g = \sqrt{2} t_0 / \sqrt{g(t_0, t_0)}$, where in a basis in which $g_{ij} = \varepsilon_i \delta_{ij}$, $\varepsilon_i = \pm 1$, and $\varepsilon_k = +1$ for some fixed $k \in \{1, \dots, n\}$ the components of t_0 are $t_0^i = \alpha \delta^{ik}$, $\alpha \in \mathbb{R} \backslash \{0\}$; so then $g(t_0, t_0) = \alpha^2 > 0$.) Now, from proposition 4.3 and corollary 4.1, case (ii), we deduce

$$\varphi^T_{q+1,p-1} \circ \varphi^T_{p,q} = \mathsf{id}_{G^U_{p,q}}, \quad \varphi^T_{p,q} \circ \varphi^T_{q+1,p-1} = \mathsf{id}_{G^U_{q+1,p-1}}. \tag{5.2}$$

from where, by virtue of [20, p. 14, proposition 6.9], the result formulated follows. □

In this way, by an explicit construction, we proved the existence of a bijective correspondence between the classes of Riemannian metrics with signature (p, q) and $(q+1, p-1)$ on any differentiable manifold admitting such metrics (and vector fields with corresponding properties - see Sect. 3). It has to be emphersized on the explicite dependence of this mapping on the choice of the vectors t_g utilized in its construction. In particular, which is essential for the physics, there is a bijective correspondence between the sets of Euclidean and Lorentzian metrics as they have signatures $(n, 0)$ and $(1, n-1)$ respectively.[9]

From here an important result follows. Since every paracompact finite-dimensional differentiable manifold admits Euclidean metrics [9, chapter IV, § 1, chapter I, example 5.7], [8, p. 280], *on any such manifold admitting a vector field with an Euclidean norm greater then one exist Lorentzian metrics* as they are in bijective correspondence with the Euclidean ones.[10] The opposite statement is also true: if on M exist Lorentzian, h, and Euclidean, e, metrics, then there is a vector field t with $e(t, t) > 1$.[11] In fact, since h is Lorentzian, there is exactly one positive eigenvalue λ_+, $\lambda_+ > 0$, for which the equation $h_{ij} t_+^j = \lambda_+ e_{ij} t_+^j$ has a non-zero solution t_+ defined up to a non-zero constant multiplier. Choosing this constant such that $h(t_+, t_+) > \lambda_+$, we find $e(t_+, t_+) > 1$. Let us recall (see Sect. 3) that the existence of t with $e(t, t) > 1$ is equivalent to the one of a non-vanishing vector field on M. So, if, as usual, we admit e, h, and t to be of class C^m, $m \geq 0$, than such a vector field may not exist on the whole M. If this happens to be the case, the above, as well as the following, considerations have to be restricted on the neighborhood(s) admitting non-vanishing vector field of class C^m.

Since (4.1) is insensitive to the change $t \mapsto -t$, we are practically dealing with the field $(t, -t)$ of linear elements, i.e. [2, sect. 2.6] a field of pairs of vector fields with opposite directions, not with the vector field t itself. If $(X, -X)$ is a field of linear elements on M, then for any $\lambda \in \mathbb{R}$, $\lambda > 1$ the vector fields $t_\pm := \pm \sqrt{\lambda / e(X, X)} X$ have Euclidean norm $e(t_\pm, t_\pm) = \lambda > 1$. Conversely, if t is a vector field with $e(t, t) > 1$, then $(t, -t)$ is a field of linear elements on M. Combining the just-obtained results, we infer that on M

[9] In the 4-dimensional case, a special type of relation between Euclidean and Lorentzian metrics is established in [7] via the Einstein equations.

[10] See [19] and [4, p.149, proposition 37] for more general results on the existens of Lorentzian metrics.

[11] Generally h, e, and t are not connected via (2.1) (see the remark at the end of Sect 5).

exist Lorentzian metrics iff on it exists a field of linear elements. This is a known result that can be found, e.g, in [2, sect. 2.6].

Let e and h be respectively Euclidean and Lorentzian metrics connected by (4.1) for some t with $e(t,t) > 1$. Now we shall prove that for a suitable choice of t the set V of vector fields on M can be split into a direct sum $V = V^+ \oplus V^-$ in which V^+ is orthogonal to V^- with respect to both e and h, and $h|_{V^\pm} = \pm\, e|_{V^\pm}$. In fact, defining $V^+ := \{t^+ : t^+ = \lambda t,\ \lambda \in \mathbb{R}\backslash\{0\}\}$ and $V^- := \{t^- : e(t^-, t) = 0\}$, we see that for $s^\pm, t^\pm \in V^\pm$ is fulfilled $e(t^-, t^+) = h(t^-, t^+) \equiv 0$, $h(s^-, t^-) \equiv -e(s^-, t^-)$ and $h(s^+, t^+) = (e(t,t) - 1)e(s^+, t^+)$. The choice of t with $e(t,t) = 2$ completes the proof. In this way we have obtained an evident special case, concerning Lorentzian metrics, of [16, p. 434, proposition VII]. As a consequence of the last proof, as well as of (4.1), we see that any set of vector fields in V^- which are mutually orthogonal (or orthonormal) with respect to e is such also with respect to h for any t with $e(t,t) > 1$ (a good choice is $e(t,t) = 2$ - see (4.9). Sets of this kind are often used in physics [2]. Evidently, if we add to such a set the victor field t, the mutual orthogonality of the vector fields of the new set will be preserved.

Another significant corollary from the proved equivalence between Lorentzian and Euclidean metrics is that any physical theory formulated in terms of (real) Lorentzian metric(s), e.g. the special theory of relativity or relativistic quantum mechanics, can equivalently be (re)formulated in terms of (real) Euclidean metric(s).[12] The price one pays for this is the introduction of an additional vector field t whose physical meaning is not a subject of this paper.

So, in some sense, the deviation of a Lorentzian metric g from an Euclidean one e can be described by an appropriate choice of certain vector field t, all connected by (2.2) under the condition $e(t,t) > 1$. In [1] this vector field is interpreted as a field of the time, the so called temporal field. In [1] on t is imposed the normalization condition $e(t,t) = 1$ (see [1, equation (3)]) which, as we proved in this paper, contradicts to the Riemannian character of the metrics considered. Consequently, this condition has to be dropped and replace with $e(t,t) > 1$. The physical interpretation of t as a temporal field will be studied elsewhere.

We also have to note that the statement in [1, p. 13] that the determinants of corresponding via (2.1) Euclidean and Lorentzian metrics differ only by sign is generally wrong. In fact, in a special basis in which (2.3) holds, we have $\det[g_{ij}] = (-1)^{n+1}(e(t,t) - 1)$ which in an arbitrary basis reads $\det[g_{ij}] = (-1)^{n+1}(e(t,t) - 1)\det[e_{ij}]$. Therefore $\det[g_{ij}] + \det[e_{ij}] = 0$ is true only in two special cases, viz. if $n = 2k$ and $e(t,t) = 2$ or if $n = 2k+1$ and $e(t,t) = 0$, $k = 0, 1, \ldots$ Moreover, by corollary 4.1, the second case cannot be realized if e is Euclidean and g Lorentzian. Thus the mentioned statement is valid only on even-dimensional manifolds and vector fields t with norm 2.

[12] For instance, in the four-dimensional case, $n = 4$, in an appropriately chosen local coordinates in which the Euclidean and Lorentzian metrics are represented respectively by the unit matrix and Minkowski metric tensor, i.e. $[e_{ij}] = \mathrm{diag}(1,1,1,1)$ and $[\eta_{ij}] = \mathrm{diag}(1,-1,-1,-1)$, the connection between them is $\eta_{ij} = t_i t_j - e_{ij}$, $t_i := e_{ij}t^j = t^i$ with $t^i = \sqrt{2}\delta^{i1}$ in the coordinates used.

6. Conclusion

The main results of the previous considerations are expressed by propositions 3.1 and 4.1–4.3. As we saw in Sect. 5, their consequence is the existence of bijective mapping between metrics of signatures (p, q) and $(q+1, p-1)$, in particular between Euclidean and Lorentzian metrics. Consequently, a physical theory formulated in terms of real Lorentzian metric(s) on a manifold admits, possibly locally, equivalent (re)formulation in terms of real Euclidean metric(s) on the same manifold. Another corollary of these propositions is that on a manifold exist metrics of signature $(q+1, p-1)$ if it admits a metric g of signature (p, q) and a vector field t with $g(t, t) > 1$. When applied to Lorentzian and Euclidean metrics, the last assertion reproduces a known result [2, sect. 2.6]. Vector fields t with $g(t, t) > 1$ exist on M iff it admits a non-vanishing vector field over the whole manifold M. If we do not impose additional conditions on the last field it always exists. But if we require it to be of class C^m with $m \geq 0$ its existence is connected with the topological properties of M and has to be explored in any particular case. Generally non-vanishing C^m vector fields exist locally, but globally this may not be the case.

Analogous consequences can be made from case (i) of corollary 4.1.

Corollary 6.1. *Let for every $g \in G^U_{p,q}$ be chosen a vector field t_g on U such that $g(t_g, t_g) = 0$ and $T := \{t_g : g \in G^U_{p,q}\}$. Then the map $\psi^T_{p,q} \colon G^U_{p,q} \to G^U_{q,p}$ defined by* (5.1) *is a bijection.*

Proof. At the beginning we notice that one can always put $t_g = 0$ for every $g \in G^U_{p,q}$ but if $p, q \geq 1$, then for any $g \in G^U_{p,q}$ exists $t_g \neq 0$ with $g(t_g, t_g) \neq 0$. (In a basis in which $g_{ij} = \varepsilon_i \delta_{ij}$, $\varepsilon_i = \pm 1$ and $\varepsilon_k + \varepsilon_l = 0$ for some fixed $k, l \in \{1, \dots, n\}$ we can set $t^i_g = \alpha(\delta^{ik} + \delta^{il})$, $\alpha \in \mathbb{R} \backslash \{0\}$.) From proposition 4.3 and corollary 4.1, case (i), we infer

$$\psi^T_{p,q} \circ \psi^T_{q,p} = \mathsf{id}_{G^U_{q,p}}, \qquad \psi^T_{q,p} \circ \psi^T_{p,q} = \mathsf{id}_{G^U_{p,q}} \tag{6.1}$$

which concludes the proof. $\square$

Hence there is bijective correspondence between metrics of signature (p, q) and (q, p). Etc. It is important to be noted that the case of a vector field t with $g(t, t) < 1$ significantly differs from the one of t with $g(t, t) \geq 1$ when some smoothness conditions are imposed: C^m, $m \geq 0$ vector field t with $g(t, t) < 1$ exists over any subset $U \subseteq M$, in particular over the whole manifold M. In fact, a trivial example of this kind is the null vector field over $U \subseteq M$.

Let us fix some bijective maps $\varphi_{p,q} \colon G^U_{p,q} \to G^U_{q+1,p-1}$ and $\psi_{p,q} \colon G^U_{p,q} \to G^U_{q,p}$ given via (5.1) for t_g with $g(t_g, t_g) = 2$ and $g(t_g, t_g) = 0$ respectively. Here $G^U_{p,q}$ is the set of Riemannian metrics on U with signature (p, q). (Let us recall that in the 'smooth' case we can not put $U = M$ as, generally, then $\varphi_{p,q}$ may not exist.) Then the map $\chi_{p,q} := \psi_{q+1,p-1} \circ \varphi_{p,q} \colon G^U_{p,q} \to G^U_{p-1,q+1}$

is bijective for any $p, q \in \mathbb{N} \cup \{0\}$ such that $p + q = n := \dim M$. Hence

$$G^U_{n,0} \xrightarrow{\chi_{n,0}} G^U_{n-1,1} \xrightarrow{\chi_{n-1,1}} G^U_{n-2,2} \xrightarrow{\chi_{n-2,2}} \cdots \xrightarrow{\chi_{2,n-2}} G^U_{1,n-1} \xrightarrow{\chi_{1,n-1}} G^U_{0,n}$$

is a sequence of bijective maps. In short, this means that there is an bijective real correspondence (given explicitly via compositions of maps like (4.1)) between Riemannian metrics of arbitrary signature. Therefore, starting from the class of Euclidean metrics on $U \subseteq M$, we can construct all other kinds of Riemannian metrics on U by means of the maps $\chi_{p,q}$, $p+q = \dim M$. Note, in the 'smooth' case the last statement may not hold globally on M but it is always valid locally.

One may ask, what would happen if the signs before the terms in the r.h.s. of (4.1) are (independently) changed? The change of the sign before the first term results in the following assertion.

Proposition 6.1. *Let g be a Riemannian metric of signature (p,q) on $U \subseteq M$, t be a vector field on U, $\widetilde{U}^-_{\substack{>\\=\\<}} := \{x |\ x \in U,\ -g(t,t)|_x \substack{>\\=\\<} 1\}$, and*

$$g \mapsto \tilde{g}^- := -g(\cdot,t) \otimes g(\cdot,t) - g. \tag{6.2}$$

Then $\tilde{g}^-$ is:

(i) *a Riemannian metric with signature (q,p) on $\widetilde{U}^-_<$.*

(ii) *a Riemannian metric with signature $(q-1,p+1)$ on $\widetilde{U}^-_>$.*

(iii) *a (parabolic) semi-Riemannian metric with signature $(q-1,p)$ and defect 1 on $\widetilde{U}^-_=$, i.e. on $\widetilde{U}^-_=$ the bilinear map h has q positive, $(p-1)$ negative, and 1 vanishing eigenvalue.*

Proof. This proof is almost identical to the one of proposition 4.1. The only difference is that in (4.2) $g(t,t)$ must be replaced by $-g(t,t)$ and in (4.3) instead of a^2 must stand $-a^2$. Formally this proof can be obtained from the one of proposition 4.1 by replacing in it t^i by $\mathrm{i}t^i$, $\mathrm{i} := \sqrt{-1}$. $\square$

The change of the sign before the second term in (4.1) and in (6.2) is equivalent to put $g = -g'$ with g' being Riemannian metric with signature (p,q). Then, since $g(t,t) = -g'(t,t)$ and the signature of g is (q,p), we obtain valid versions of corollary 4.1 and proposition 6.1 if we replace in them g, p, and q with $-g$, q, and p respectively. Thus we have proved:

Proposition 6.2. *Let g be a Riemannian metric of signature (p,q) on $U \subseteq M$, t be a vector field on U, $U^\pm_{\substack{>\\=\\<}} := \{x |\ x \in U,\ \mp g(t,t)|_x \substack{>\\=\\<} 1\} = \widetilde{U}^\mp_{\substack{>\\=\\<}}$, and*

$$g \mapsto g^\pm := \pm g(\cdot,t) \otimes g(\cdot,t) + g. \tag{6.3}$$

Then $g^\pm$ is:

(i) *a Riemannian metric with signature (p,q) on $U^\pm_<$.*

(ii) *a Riemannian metric with signature $(p \pm 1, q \mp 1)$ on $U^\pm_>$.*

(iii) *a (parabolic) semi-Riemannian metric with defect 1 and signature $(p + (\pm 1 - 1)/2, q + (\mp 1 - 1)/2)$ on $U^\pm_=$, i.e. in this case $g^\pm$ has $p + (\pm 1 - 1)/2$ positive, $q + (\mp 1 - 1)/2$ negative, and 1 vanishing eigenvalue.*

For the metrics $g^{\pm}$ and $\tilde{g}^{-}$ can be proved analogous results as those for $\tilde{g}^{+} := h$ in sections 4 and 5. Since this is an almost evident technical task, we do not present them here. In connection with this, we will note only that the equalities $\widetilde{(\tilde{g}^{\pm})}^{\pm} = g$ and $(g^{\pm})^{\mp} = g$ are valid iff $\pm g(t,t) = 0, +2$ and $\pm g(t,t) = 0, -2$ respectively, while the equations $\widetilde{(\tilde{g}^{\pm})}^{\mp} = g$ and $(g^{\pm})^{\pm} = g$ can not be fulfilled for (real) Riemannian metrics as they are equivalent to $\pm g(t,t) = 1 - \mathrm{i}, 1 + \mathrm{i}$ and $\pm g(t,t) = -1 - \mathrm{i}, -1 + \mathrm{i}$ respectively, $\mathrm{i} := +\sqrt{-1}$.

Metrics like $g^{\pm}$ find applications in exploring modifications of general relativity. For instance, up to a positive real constant, the defined in [7, sect. IV, equation (4.1)] metric g^{Einst} is of the type g^{ε} with $\varepsilon = \operatorname{sign}(-\lambda)$ and $t_i = \sqrt{|2\lambda|}\eta_i$ with $\lambda := \frac{\alpha+\beta}{\alpha+2\beta}$, where the real parameters α and β and the covector η_i are described in [7, sect. II].

A corollary of proposition 6.2 is the assertion of [2, sect. 2.6], [3, p. 219], and [4, p.149, lemma 36] that if g is an Euclidean metric and X is non-zero vector field, then $h = g - 2g(\cdot, X) \otimes g(\cdot, X)/g(X, X)$ is a Lorentzian metric. In fact, putting $t = \sqrt{2}X/\sqrt{g(X,X)}$ $(= \sqrt{2}U$ in the notation of [4]), we get $h = g - g(\cdot, t) \otimes g(\cdot, t)$ and $g(t,t) = -2 < -1$. Therefore h has signature $(n-1, 1)$ as that of g is $(n, 0)$, i.e. it is a Lorentzian metric according to the accepted in [2, 3, 4] definition.

There is a simple, but useful for the physics result. Given metrics g, $g^{\pm}$, and $\tilde{g}^{\pm}$ and a vector field t non-isotropic with respect to g (i.e. $g(t,t) \neq 0$), all connected via (4.1), (6.2), and (6.3). Then there exist (local) fields of bases orthogonal with respect to all these metrics. To prove this, we notice that if $\{E_i\}$ is a field of bases with $E_n = \lambda t$, $\lambda \neq 0, \infty$ and $g(E_i, E_j) = \alpha_i \delta_{ij}$, where $\alpha_i : M \to \mathbb{R}\backslash\{0\}$ and δ_{ij} are the Kroneker δ-symbols, then $g^{\pm}(E_i, E_j) = \beta_i^{\pm}\delta_{ij}$, where $\beta_i^{\pm} = \alpha_i$ for $1 \leq i < n$ and $\beta_n^{\pm} = \alpha_n \pm \alpha_n^2/\lambda^2$, and $\tilde{g}^{\pm}(E_i, E_j) = \tilde{\beta}_i^{\pm}\delta_{ij}$ with $\tilde{\beta}_i^{\pm} = -\alpha_i$ for $1 \leq i < n$ and $\tilde{\beta}_n^{\pm} = -\alpha_n \pm \alpha_n^2/\lambda^2$.

We end with the remark that the results of this paper may find possible applications in the theories based on space-time models with changing signature (topology) (see, e.g. [21, 22]).

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